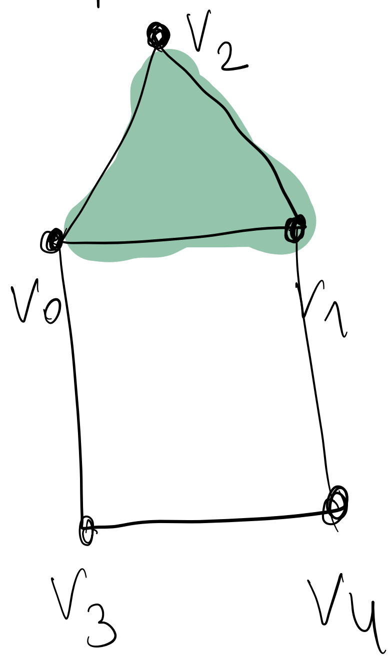
Definition

When $C_n = \Delta_n(X)$, the homology group $\ker \partial_n / \operatorname{Im} \partial_{n+1}$ is denoted by $H_n^{\Delta}(X)$ and called the **nth simplicial homology group of X.**

Intuition: $H_n^{\Delta}(X)$ captures the information about n-dim holes in X.

Example 1 Homology groups of X

X



$\Delta_0(X) = \langle v_0, v_1, v_2, v_3, v_4 \rangle$

$\Delta_1(X) = \langle [v_0, v_2], [v_0, v_1], [v_0, v_3], [v_1, v_2], [v_1, v_4], [v_3, v_4] \rangle$

$\Delta_2(X) = \langle [v_0, v_1, v_2] \rangle$

$[v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$\Delta_3(X) = 0$

Chain complex associated to X

$$\to 0 \to \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{0} 0$$

$$\partial_2([v_0, v_1, v_2]) =$$
$$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$\mathrm{Ker}\,\partial_2$ is trivial, so $H_2(X) = 0$.

Also $H_i(X) = 0$ for $i \geq 3$.

Kernel of ∂_1:

$$\partial_1[v_0, v_2] = v_2 - v_0$$
$$\partial_1[v_0, v_1] = v_1 - v_0$$
$$\partial_1[v_0, v_3] = v_3 - v_0$$
$$\partial_1[v_1, v_2] = v_2 - v_1$$
$$\partial_1[v_1, v_4] = v_4 - v_1$$
$$\partial_1[v_3, v_4] = v_4 - v_3$$

$$a(v_2-v_0)+b(v_1-v_0)+c(v_3-v_0)+d(v_2-v_1)+$$
$$e(v_4-v_1)+f(v_4-v_3)=0$$

$$v_0\overset{①}{(-a-b-c)}+v_1\overset{②}{(b-d-e)}$$
$$+v_2\overset{③}{(a+d)}+v_3\overset{④}{(c-f)}+v_4\overset{⑤}{(e+f)}=0$$

4 $f=c$
3 $d=-a$
5 $e=-f=-c$

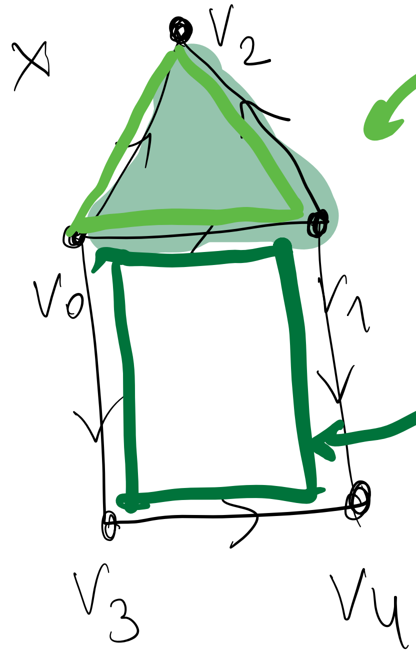
1 $b-d-e=0$

$$b=d+e=$$
$$=-a-c$$

2 $(-a-c)-(-a)-(-c)=0$

$$(a,-a-c,c,-a,-c,c)=$$
$$=a(1,-1,0,-1,0,0)+c(0,-1,1,0,-1,1)$$

$$\ker \partial_1 = \langle [v_0,v_2]-[v_0,v_1]-[v_1,v_2],$$
$$-[v_0,v_1]+[v_0,v_3]-[v_1,v_4]+[v_3,v_4] \rangle$$



$[v_0,v_2]-[v_0,v_1]-[v_1,v_2]$

$-[v_0,v_1]+[v_0,v_3]-[v_1,v_4]+[v_3,v_4]$

$$\operatorname{Im} \partial_2 = [v_1,v_2]-[v_0,v_2]+[v_0,v_1]$$

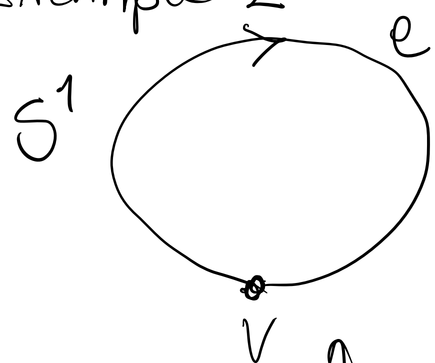
$$H_1^\Delta(X) = \ker \partial_1 / \operatorname{Im} \partial_2 \cong \langle [v_0,v_1]+[v_0,v_3]-[v_1,v_4]+[v_3,v_4] \rangle$$

$$\cong \mathbb{Z}$$

the space X has one hole

This is an example of a special type of a Δ-complex, called a **SIMPLICIAL COMPLEX**. Simplicial complexes can be encoded combinatorially and software exists to compute their homology groups (coefficients are taken from a finite field, so that the computation reduces to linear algebra).

Example 2



Δ-complex structure on S^1

$$\Delta_0(S^1) = \langle v \rangle$$
$$\Delta_1(e) = \langle e \rangle$$
$$\Delta_i(x) = 0 \quad \forall\, i \geq 2$$

$$\Delta_1(e) \xrightarrow{\partial_1} \Delta_0(S^1) \to 0 \qquad \partial_1(e) = v - v = 0$$

chain complex associated to S^1

$$H_1^{\Delta}(X) = \ker \partial_1 = \langle e \rangle \cong \mathbb{Z}$$

$$H_0^{\Delta}(X) = \Delta_0(S^1) / \operatorname{Im}\partial_1 \cong \langle v \rangle \cong \mathbb{Z}$$

$$H_i^{\Delta}(X) = 0 \text{ for } i \geq 2$$

Questions: Are the groups $H_n^{\Delta}(X)$ independent of the choice of Δ-complex structure on X?

If two Δ-complexes are homeomorphic ($\cong$), do they have isomorphic homology groups?

To answer these, we first develop a more general theory of singular homology groups, which have the added benefit of being defined for all spaces, not just Δ-cx.